It follows from (2.1), (3.4) and (3.7) that in that case $R^\circ - H^\circ \to 0$ and the separation boundary becomes the straight line $y = y_2(x) = -H$, i.e. that boundary becomes an impermeable base.

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OPTIMUM CONTROL OF THE FORCED MOTIONS OF SYSTEMS WITH CONTINUOUSLY DISTRIBUTED PARAMETERS

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We propose one of the possible versions of the optimum control of the forced motions of elastic systems of the type of rods, plates, and shells. We apply the procedure developed to elementary problems on the transition of a freely-supported rod or plate from an initial state φ, ψ to the rest state in the least possible time T in the presence of a constraint on the forcing load. We use the elementary results of theory of the l-problem of moments of Krein [1 - 3].

1. We consider a hinge-supported rod undergoing forced motions under the action of a load $f(x, t)$. The complete system of equations defining the state of the rod at any instant t has the form

$$\frac{\partial^4 w}{\partial x^4} + \frac{\rho F}{EJ}\frac{\partial^2 w}{\partial t^2} = \frac{f(x,t)}{EJ}, \quad x \in (0, l), \quad t > 0$$
$$w(0,t) = w(l,t) = 0, \quad w_{xx}(0,t) = w_{xx}(l,t) = 0 \tag{1.1}$$
$$w(x,0) = \varphi(x), \ w^{\cdot}(x,0) = \psi(x), \ w^{\cdot} = \partial w / \partial t$$

Here $w(x, t)$ is the vertical displacement, ρ and E are the density and the modulus of elasticity of the material, J and F are the moment of inertia and the area of the rod's cross section. We represent the forcing load (the control) in the following way:

$$f(x, t) = \sum_{k=1}^{\infty} f_k(t) \sin \frac{k\pi x}{l}, \quad 0 \leqslant t < T \tag{1.2}$$
$$f(x, t) = 0, \quad t \geqslant T$$

The system $\{f_k(t)\}$ is assumed to be linearly independent.

The solution of problem (1.1) has the form

$$w(x, t) = \sum_{k=1}^{\infty} \left\{ C_k \sin \lambda_k t + D_k \cos \lambda_k t + \frac{1}{\lambda_k \rho F} \int_0^t f_k(\eta) \sin \lambda_k (t - \eta)\, d\eta \right\} \sin \frac{k\pi x}{l} \tag{1.3}$$

$$D_k = \frac{2}{l} \int_0^l \varphi(\xi) \sin \frac{k\pi\xi}{l}\, d\xi, \qquad 0 \leqslant t \leqslant T$$

$$C_k = \frac{2}{l\lambda_k} \int_0^l \psi(\xi) \sin \frac{k\pi\xi}{l}\, d\xi, \qquad \lambda_k = \left(\frac{k\pi}{l}\right)^2 \sqrt{\frac{EJ}{\rho F}}$$

By interpreting the $f_k(t)$ as control functions, we pose the problem of "damping" the rod, i.e. of transferring it from a state $\varphi(x)$, $\psi(x)$ to the state of rest in a minimal time T. We assume that the controls $f_k(t)$ belong to the space L^p $(1 \leqslant p \leqslant \infty)$ of functions summable to the pth power on $[0, T]$. On the $f_k(t)$ we impose the constraints

$$\|f_k\|_{L^p} \leqslant \Lambda, \qquad k = 1, 2, \ldots \tag{1.4}$$

The transfer conditions for the system from the initial state to the final have the form

$$w|_{t=T} = 0, \quad w^{\cdot}|_{t=T} = 0, \quad x \in [0, l] \tag{1.5}$$

By substituting the function (1.3) into (1.5), we arrive at the system of relations

$$C_k \sin \lambda_k T + D_k \cos \lambda_k T + \frac{1}{F\rho\lambda_k} \int_0^T f_k(\eta) \sin \lambda_k (T - \eta)\, d\eta = 0 \tag{1.6}$$

$$C_k \cos \lambda_k T + D_k \sin \lambda_k T + \frac{1}{F\rho\lambda_k} \int_0^T f_k(\eta) \cos \lambda_k (T - \eta)\, d\eta = 0$$

$$k = 1, 2, \ldots$$

By eliminating the time T from the integrands in relations (1.6), we arrive at a denumerable system of second-order moment problems

$$a_k = \int_0^T f_k(\eta) \cos \lambda_k \eta\, d\eta, \qquad a_k = -C_k \rho F \lambda_k \tag{1.7}$$

$$b_k = \int_0^T f_k(\eta) \sin \lambda_k \eta\, d\eta, \qquad b_k = D_k \rho F \lambda_k, \quad \|f_k\| \leqslant \Lambda, \quad f_k \in L^p, \quad 1 \leqslant p \leqslant \infty$$

Each of the indicated collection of moment problems (1.7) has its own minimal time T_k. The largest of $\{T_k\}$ (if it exists) yields the least possible time T in which the transition of the rod from the state φ, ψ to the state of rest in the presence of constraints (1.3) is realized.

2. According to [1–3], each of the moment problems (1.7) reduces to the following equivalent problem: find

$$\min_{\xi_k, \eta_k} \int_0^T |\xi_k \cos \lambda_k t + \eta_k \sin \lambda_k t|^q dt = \tag{2.1}$$

$$\int_0^T |\xi_k^\circ \cos \lambda_k t + \eta_k^\circ \sin \lambda_k t|^q dt \geqslant \Lambda^{-q}, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

under the condition

$$\xi_k a_k + \eta_k b_k = 1 \quad (k \text{ is fixed})$$

The conditional extremum problem (2.1) is solved elementarily. By assuming for the sake of shortening the writing that $a_k = 0$ (e.g. the initial and final velocities of the rod equal zero) and that $f_k \in L^2$, we obtain an expression for the quantities ξ_k° and η_k° $(p = q = 2)$

$$\eta_k = \eta_i \quad \frac{1}{b_k}, \quad \xi_k^\circ = \frac{\cos 2\lambda_k T - 1}{2\lambda_k T + \sin 2\lambda_k T} \frac{1}{b_k} \tag{2.2}$$

Substituting (2.2) into the integrand in (2.1), integrating with $q = 2$ and equating the result obtained to the quantity Λ^{-2}, we arrive at a transcendental equation in the least possible time $T = T_k$ in each of problems (2.1)

$$1 - \frac{\sin^2 \tau_k}{\tau_k^2} = \frac{\varkappa_k}{\tau_k}\left(1 + \frac{\sin 2\tau_k}{2\tau_k}\right) \tag{2.3}$$

$$\varkappa_k = \frac{2\lambda_k b_k^2}{\Lambda^2}, \qquad \tau_k = \lambda_k T_k$$

T h e o r e m 2.1. For any value $m\pi < \varkappa < (m+1)\pi$, $(m = 0, 1, \ldots)$ there exists a unique solution δ of the equation

$$1 - \frac{\sin^2 \delta}{\delta^2} = \frac{\varkappa}{\delta}\left(1 + \frac{\sin 2\delta}{2\delta}\right) \tag{2.4}$$

contained in the interval $m\pi < \delta < (m+1)\pi$. When $\varkappa = m\pi$ the solution $\delta = m\pi$. As $\varkappa$ increases, the function $\delta(\varkappa)$ increases monotonically on $(m\pi, m\pi + \pi)$, $\lim \delta / \varkappa = 1$ as $\varkappa \to \infty$.

P r o o f. We introduce the function

$$R(\delta) = \delta - \lambda r(\delta), \qquad r(\delta) = \frac{\delta^2 - \sin^2 \delta}{2\delta + \sin 2\delta} - \frac{\varkappa}{2} \tag{2.5}$$

$$r'(\delta) \geqslant 0, \qquad \lambda = \{\max r'(\delta)\}^{-1}, \qquad m\pi \leqslant \delta \leqslant (m+1)\pi,$$

$$r(m\pi) < 0, \; r(m\pi + \pi) > 0$$

Obviously, the equation $\delta = R(\delta)$ is equivalent to (2.4). By virtue of (2.5) $|R'(\delta)| < 1$ on $(m\pi, m\pi + \pi)$, i.e. R – contraction mapping. By the contraction mapping principle [4] this equation has a unique solution $m\pi < \delta < (m+1)\pi$. The monotonic growth of $\delta(\varkappa)$ on each of the intervals $(m\pi, m\pi + \pi)$ follows from the fact that the derivative $d\delta / d\varkappa$ is positive on it.

From the proof it follows, in particular, that the solution of Eq. (2.4) can be obtained by the method of successive approximations by a recurrence process:

$$\delta_{n+1} = R(\delta_n), \quad n = 0, 1, \ldots; \quad \delta = \lim_{n\to\infty} \delta_n \tag{2.6}$$

The roots of Eq. (2.4) as a function of the values of parameter $\varkappa$ are presented below:

$\varkappa = 0.005$,	0.010,	0.015,	0.020,	0.025,	0.5,
$\delta = 0.308$,	0.387,	0.441,	0.484,	0.522,	1.308,
$\varkappa = 1$,	2,	10,	15,	20,	30,
$\delta = 1.603$,	2.017,	10.48,	14.625,	20.23,	29.94

Assume that $\varphi(x) \in C^n$ $(n \geqslant 1)$, while $d^n\varphi / dx^n$ is of bounded variation on $[0, l]$. Under these conditions the following theorem is valid.

Theorem 2.2. The problem of the optimal (in the sense indicated above) control of the rod in L^2 is solvable. The least possible time T in which the rod can be transferred from the initial state to the final (in the presence of constraints (1.4)) is determined by the formula

$$T = \sup\left\{T_k = \frac{\tau_k}{\lambda_k}, \quad k = 1, 2, \ldots\right\} = \frac{\tau_{k_0}}{\lambda_{k_0}} = T_{k_0} \tag{2.7}$$

Proof. By virtue of the conditions imposed on $\varphi(x)$, the Fourier coefficients D_k decrease with the increase of k no slower than k^{-3}. On the basis of (1.3), (1.7) and (2.3) we conclude that the sequence $\{\varkappa_k\}$ is bounded. By virtue of Theorem 2.1 the sequence $\{\tau_k = \tau(\varkappa_k)\}$ also is bounded. Hence follows the existence of a finite least upper bound of the sequence $\{\tau_k / \lambda_k\}$, and it is realized for some $k = k_0$. Thus, (2.7) makes sense and defines the least possible time to which there correspond simultaneously the solutions of the whole collection of moment problems (1.7).

Further, series (1.2), setting the controlling load $f(x, t)$, converges. In fact, by virtue of (2.2) the desired control functions $f_k(t)$ have in L^2 the following form [3]:

$$f_k(t) = \frac{1}{\beta_k} G(\xi_k^\circ, \eta_k^\circ), \quad G(\xi_k, \eta_k) = \xi_k \cos\lambda_k t + \eta_k \sin\lambda_k t \tag{2.8}$$

$$\beta_k = \int_0^T |G(\xi_k^\circ, \eta_k^\circ)|^2\, dt \geqslant \Lambda^{-2}, \quad \beta_{k_0} = \Lambda^{-2}, \quad k = 1, 2, \ldots$$

$$\xi_k^\circ = \frac{1}{b_k} \frac{\cos 2\varepsilon_k\tau^\circ - 1}{2\varepsilon_k\tau^\circ + \sin 2\varepsilon_k\tau^\circ}, \quad \eta_k^\circ = \frac{1}{b_k}$$

$$\varepsilon_k = \frac{\lambda_k}{\lambda_{k_0}} = \left(\frac{k}{k_0}\right)^2, \quad \tau^\circ = \tau_{k_0}$$

Allowing for (1.3) and (1.7), from (2.7) we immediately find

$$f_k(t) = \frac{2EJk_0^2}{\tau^\circ}\left(\frac{\pi}{l}\right)^4 k^2 D_k \{A_k(\tau^\circ)\sin\lambda_k t - B_k(\tau^\circ)\cos\lambda_k t\} \tag{2.9}$$

$$A_k(\tau^\circ) = \left(1 + \frac{\sin 2\varepsilon_k\tau_0}{2\varepsilon_k\tau^\circ}\right)\frac{1}{\Delta_k(\tau^\circ)}, \quad k = 1, 2, \ldots$$

$$B_k(\tau^\circ) = \frac{\sin^2 \varepsilon_k\tau^\circ}{\varepsilon_k\tau^\circ\Delta_k(\tau^\circ)}, \quad \Delta_k(\tau^\circ) = 1 - \left(\frac{\sin \varepsilon_k\tau^\circ}{\varepsilon_k\tau^\circ}\right)^2$$

The convergence in the norm in L^2 of series (1.2) for any fixed $0 \leqslant t \leqslant T_{k_0}$ follows from (2.9). However, if $n \geqslant 2$, series (1.2) converges absolutely and uniformly on $[0, l]$ for any $0 \leqslant t \leqslant T_{k_0}$.

3. Let us consider some examples. Let the initial velocity $\psi(x) = 0$, while the initial displacement coincides with the flexure of the rod under the action of a uniform transverse load $s = \text{const}$, i.e.

$$\varphi(x) = \sum_{k=1}^{\infty} D_k \sin\frac{k\pi x}{l}, \qquad D_k = \frac{2s(1-\cos k\pi)}{EJl}\left(\frac{l}{k\pi}\right)^5 \tag{3.1}$$

Let us determine the controlling load $f(x, t)$ and the least possible time T needed to dampen the rod. The control functions $f_k(t)$ are sought in L^2.

In the given case, obviously, $f_{2k}(t) = 0$, $k = 1, 2, \ldots$. By virtue of (1.7), (1.3), (2.3), (2.7) and (2.9), the quantities $\varkappa_k$ and β_k have the form

$$\varkappa_k = \frac{32s^2l^2}{\pi^4k^4\Lambda^2}\sqrt{\frac{\rho E}{EJ}}$$
$$\beta_k = \frac{(k^2\tau^\circ)^2 - (\sin k^2\tau^\circ)^2}{2k^2\tau^\circ + \sin 2k^2\tau^\circ}\,\frac{1}{\lambda_k b_k^2}, \qquad k = 1, 3, \ldots \tag{3.2}$$

By virtue of Theorems 2.2 and 2.1 the solution of the optimal damping problem of the rod exists and the least possible control time is

$$T = T_1 = \tau_1 / \lambda_1, \qquad \tau^0 = \tau_1 \tag{3.3}$$

We find the control functions from (2.9) and (3.1) with $k_0 = 1$

$$f_k(t) = \frac{8S}{\pi\tau^0 k^3}\{A_k(\tau^\circ)\sin\lambda_k t - B_k(\tau^\circ)\cos\lambda_k t\}, \quad 0 \leqslant t \leqslant T_1 \tag{3.4}$$

We find the controlling load $f(x, t)$ by formula (1.2) into which we must substitute the functions $f_k(t)$ from (3.4).

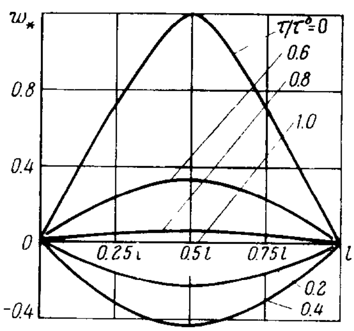

Fig. 1

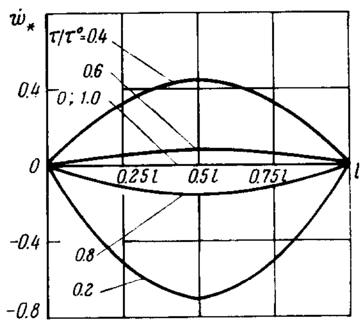

Fig. 2

In Figs. 1–3 we have shown the curves $w_* = \pi^5 EJw / 4sl^4$, $w_*^\cdot = \pi^3\sqrt{EJ\rho E}\, w^\cdot / 4sl^2$ and $f(x, t) / s$ for various values of the dimensionless time $\tau = \lambda_1 t$ $(0 \leqslant \tau \leqslant \tau^0)$ for $\tau^0 = 10$.

As a second example we consider the rod damping problem for $\psi(x) = 0$ and an initial displacement coinciding with the flexure of the rod under the action of a point force P applied at the middle. In this case we have

$$\varkappa_k = \frac{8P^2}{\pi^2 k^2 \Lambda^2} \sqrt{\frac{\rho F}{EJ}}, \qquad D_k = \frac{2P}{EJl}\left(\frac{l}{k\pi}\right)^4 \sin\frac{k\pi}{2}$$

Consequently, the least possible control time agrees with T_1 and $\tau^0 = \tau_1$. The control functions have the form

$$f_k(T) = \frac{4P \sin k\pi/2}{k^2 l \tau^\circ}(A_k \sin\lambda_k t - B_k \cos\lambda_k t), \qquad 0 \leqslant t < T$$

The quantities A_k and B_k are given in (3.4). The controlling load $f(x, t)$ is determined by series (1.2).

Let us now consider the case, most severe from the viewpoint of optimal damping of

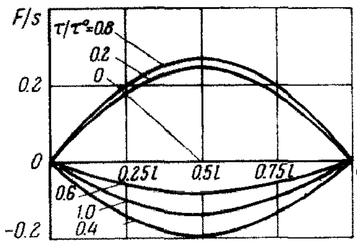

Fig. 3

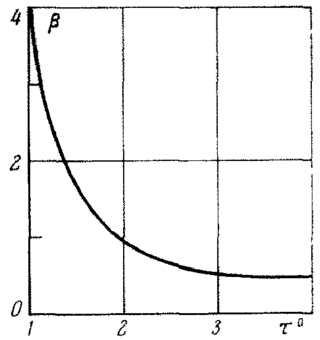

Fig. 4

the rod, when the initial displacement coincides with the rod's flexure under the action of a bending moment M concentrated at a point $x = x_0$. Here

$$\varkappa_k = \frac{8M^2}{\Lambda^2 l^2}\sqrt{\frac{\rho F}{EJ}}\cos^2\frac{k\pi x_0}{l}, \qquad D_k = \frac{2M}{EJl}\left(\frac{l}{k\pi}\right)^3 \cos\frac{k\pi x_0}{l}, \quad C_k = 0$$

When $x_0 = 0.5\ l$, for example, the minimal damping time $T = T_1$ ($\tau^\circ = \tau_1$). The control functions are:

$$f_k(t) = \frac{2\pi M}{k l^2 \tau^\circ}(A_k \sin\lambda_k t - B_k \cos\lambda_k t)\cos\frac{k\pi}{2}, \qquad \|f_k\| \leqslant \Lambda$$

Here A_k and B_k have been defined in (3.4).

4. Let us connect the constraint Λ with the static load necessary to create the initial flexure $\varphi(x)$. We present our arguments with reference to the first example in Sect. 3. Taking into account that $\|s\| = s\sqrt{T}$ in $L^2\,[0, T_2]$ from (3.2), (3.3) and (1.3) we find

$$\beta \quad \frac{\Lambda}{s\sqrt{T}} = \frac{4}{\pi}\sqrt{\frac{2}{\varkappa\tau^\circ}}$$

The curve $\beta(\tau^0)$ is shown in Fig. 4. The calculation can be made in the following order. The quantity β and the rod parameters l, ρ, F and J are prescribed. We determine τ° from the curve $\beta(\tau^\circ)$ in Fig. 4. We find the damping time T by formula (3.3), while the ratio of the controlling load $f(x, t)$ to the intensity s of the static load, by formula (1.2) with due regard to (3.4). The quantity s is fixed since the initial flexure of the rod is specified. Analogous arguments are carried out in the other cases.

5. Let us consider the optimal damping problem for a rectangular plate hinge-supported along the contour. The complete system of equations defining the forced motions of the plate has the form

$$\nabla^2\nabla^2 w(x, y, t) + \frac{\rho h}{D}\frac{\partial^2 w}{\partial t^2} = \frac{1}{D} f(x, y, t) \tag{5.1}$$

$$0 < x < a, \quad 0 < y < b, \quad t > 0$$

$$w(x, y, t)|_c = 0, \quad w_{nn}(x, y, t)|_c = 0, \quad t \geqslant 0$$

$$w(x, y, t)|_{t=0} = \varphi(x, y), \quad w^{\cdot}(x, y, t)|_{t=0} = \psi(x, y)$$

$$D = Eh^3/12(1 - \mu^2)$$

Here $w(x, y, t)$ is the plate's vertical displacement; ρ, E, μ are the density, the modulus of elasticity and the Poisson's ratio, respectively, h is the plate's thickness; c is the boundary of the rectangular region B $(0 \leqslant x \leqslant a, \ 0 \leqslant y \leqslant b)$. By the sense of the problem the functions φ and ψ are continuous, possess continuous first and bounded second derivatives in all their arguments in B. The controlling load is sought in the form

$$f(x, y, t) = \sum_{m,n=1}^{\infty} f_{mn}(t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{5.2}$$

$$0 \leqslant t < T$$

$$f(x, y, t) = 0, \quad t \geqslant T$$

Then the solution of problem (5.1), corresponding to load (5.2), is represented in the usual form

$$w(x, y, t) = \sum_{m,n=1}^{\infty} \Big\{ C_{mn} \sin\lambda_{mn} t + D_{mn} \cos\lambda_{mn} t + \tag{5.3}$$

$$\frac{1}{\rho h \lambda_{nm}} \int_0^t f_{mn}(\eta) \sin\lambda_{mn}(t - \eta)\, d\eta \Big\} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$D_{mn} = \frac{4}{ab} \int_0^a \int_0^b \varphi(x, y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

$$C_{mn} = \frac{4}{ab\lambda_{mn}} \int_0^a \int_0^b \psi(x, y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

$$\lambda_{mn} = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \sqrt{\frac{D}{\rho h}}$$

Starting from solution (5.3) we pose the problem of transferring the plate from a state $\varphi(x, y)$, $\psi(x, y)$ to the state of rest in the least possible time T. The conditions for the total damping of the plate have the form

$$w|_{t=T} = 0, \quad w^{\cdot}|_{t=T} = 0, \quad 0 \leqslant x \leqslant a, \quad 0 \leqslant y \leqslant b \tag{5.4}$$

Substituting function (5.3) with $t = T$, into (5.4), we obtain (after intermediate manipulations) a denumerable collection of second-order moment problems

$$a_{mn} = \int_0^T f_{mn}(\eta) \cos\lambda_{mn}\eta\, d\eta, \quad \|f_{mn}\| \leqslant \Lambda$$

$$b_{mn} = \int_0^T f_{mn}(\eta) \sin\lambda_{mn}\eta\, d\eta, \quad f_{mn} \in L^p$$

$$1 \leqslant p \leqslant \infty$$

$$a_{mn} = -\rho h \lambda_{mn} C_{mn}, \quad b_{mn} = \rho h \lambda_{mn} D_{mn}, \quad m, n = 1, 2, \ldots$$

Thus, we arrive at the problem studied above.

It is obvious that the procedure described can be applied also to the optimal control problems for the forced motions of elastic shells.

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ON THE REISSNER-NAGHDI ELASTICITY RELATIONSHIPS

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Shell theory equations are constructed by the method in [1] to the accuracy of quantities of the order of $h_*^{2+\kappa}$, where $k = 0$ for $0 \leqslant t \leqslant 1/2$ and $k = 2-4t$ for $1/2 \leqslant t < 1$ (h_* is the relative semithickness of the shell and t is the index of the state of stress variation). Without being within the framework of the Love-type theory, the equations obtained are compared with the Reissner-Naghdi equations [2, 3] in which the transverse shear is taken into account, and it is shown that from the asymptotic viewpoint these latter are inconsistent. It is also shown that if the shell resists shear weakly, then from the asymptotic viewpoint the Reissner-Naghdi theory is completely well founded.

The three-dimensional equations of elasticity theory are reduced to two-dimensional equations in [1] by using an asymptotic method, i.e. all members of the same order relative to the small parameter h_* are taken into account at each stage of the calculations. It has been shown that without going outside the framework of the ordinary concepts of the Love-type theory of shells (in particular, without taking account of transverse shear), the shell theory equations can be constructed to the accuracy of quantities of the order of h_*^{2-2t}, but it is impossible to exceed this limit without a qualitative complication in the theory.

1. To construct a shell theory to the accuracy of quantities of the order of h_*^{2+k} ($k = 0$ for $t \leqslant 1/2$ and $k = 2-4t$ for $1/2 < t < 1$) let us use the asymptotic representation of the quantities in three-dimensional elasticity theory used in [1].

The terminology and notation used henceforth correspond to that used in [1, 4].

Let us take the equations of three-dimensional elasticity theory referred to a tri-ortho-